

# Flows in Networks: The Ford-Fulkerson Algorithm

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## Learning Objectives

- Compute maximum flows in networks.

# Algorithm

Idea:

- Start with zero flow.
- Repeatedly add flow.
- Stop when you cannot add more.

# Adding Flow

- Have flow  $f$ .
- Compute residual  $G_f$ .
- Any new flow  $f + g$ , where  $g$  a flow for  $G_f$ .
- Need to find flow for  $G_f$ .
- See if there's a source-sink path.

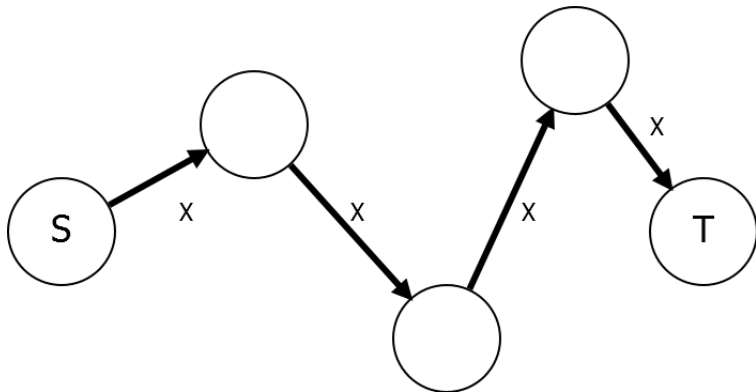
# No Path

If there's no source-sink path in  $G_f$ :

- Reachable vertices define cut of size 0.
- No  $g$  of positive size.
- $|f + g| \leq |f|$ .
- $f$  is a maxflow.

# Path

If there is a path, add flow along path.



Need  $X \leq \min_{e \in \text{path}} C_e$ .

# Adding Flow

- Find flow  $g$  for  $G_f$  with  $|g| > 0$ .
- Replace  $f$  by  $f + g$ .
- $|f + g| > |f|$ .

# Pseudocode

## Ford-Fulkerson( $G$ )

$f \leftarrow 0$

repeat:

    Compute  $G_f$

    Find  $s - t$  path  $P$  in  $G_f$

    if no path: return  $f$

$X \leftarrow \min_{e \in P} C_e$

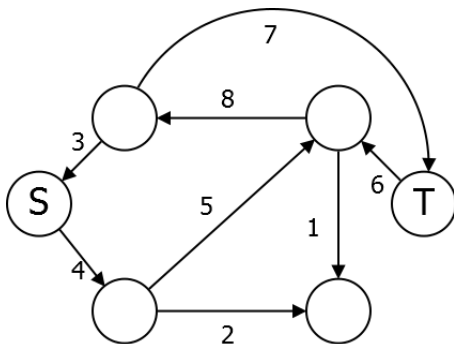
$g$  flow with  $g_e = X$  for  $e \in P$

$f \leftarrow f + g$



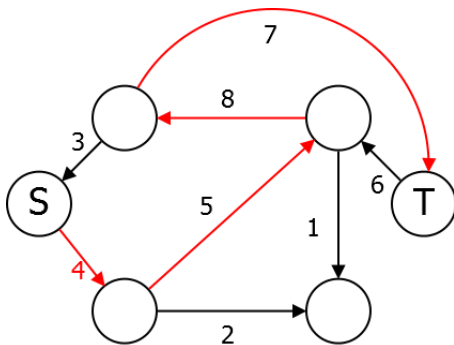
# Problem

How much flow is added in one step?

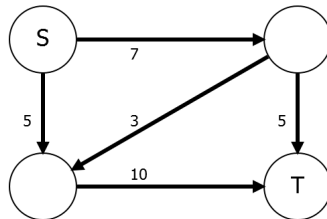
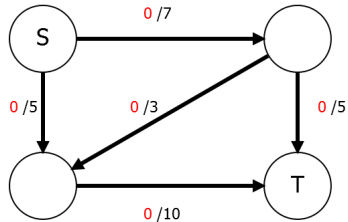


# Solution

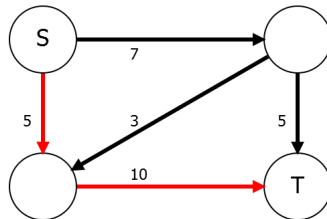
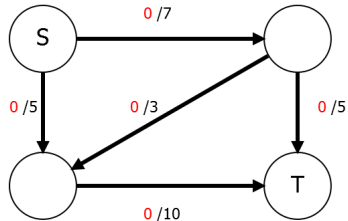
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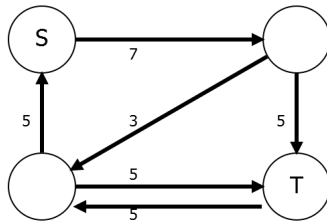
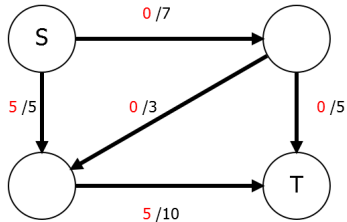
# Example



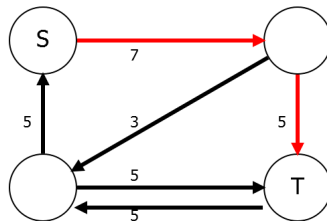
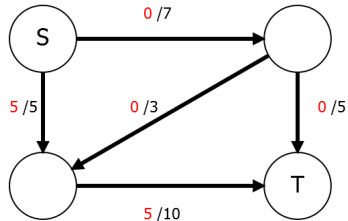
# Example



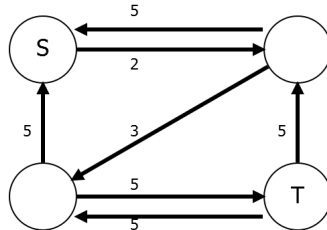
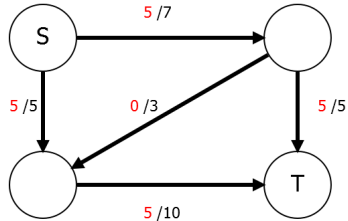
# Example



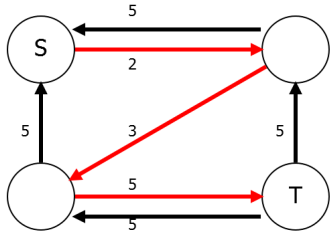
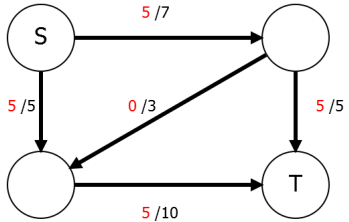
# Example



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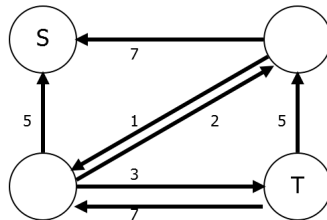
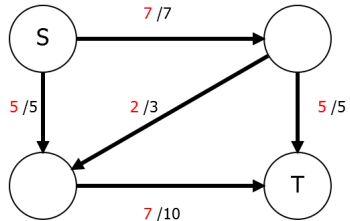


# Example

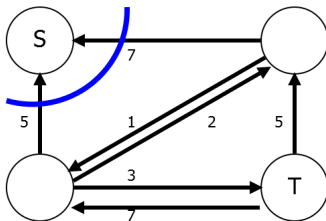
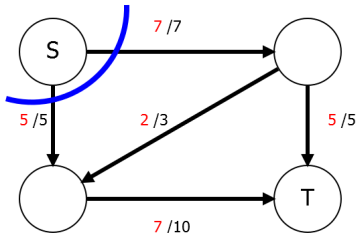




# Example



# Example



# Integrality

Note that if all capacities are integers, all flows we produce are integral.

## Lemma

If the network  $G$  has integral capacities, there is a maximum flow with integral flow rates.

# Analysis

[Assume integral capacities]

- Can compute  $G_f$  and find  $P$  in  $O(|E|)$  time.
- Each time, increase total flow by at least 1.
- Total runtime:  $O(|E||f|)$ .

# Analysis

[Assume integral capacities]

- Can compute  $G_f$  and find  $P$  in  $O(|E|)$  time.
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Note: Potentially quite large if flow is numerically large.

# Non-Determinacy

Note that the algorithm says to find **an**  $s - t$  path in  $G_f$ . There might be many valid paths to choose from. Using DFS is fast, but perhaps not the best. As we will see the way we pick our path will affect the runtime of the algorithm.