

Flows in Networks: Network Flows

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Advanced Algorithms and Complexity
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Learning Objectives

- Provide the definitions of a network and a flow.
- Give some examples of real world situations in which network flow problems might arise.

Last Time

- Last time: Discussed disaster management problem.
- Today: Talk about formalization of this and similar problems.

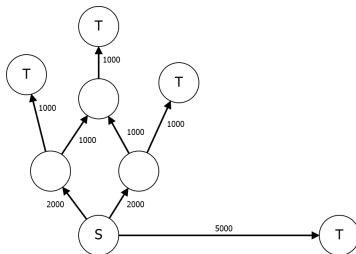
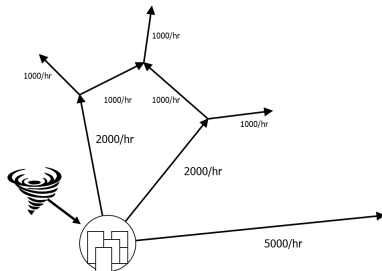
Network

Definition

A **network** is a directed graph G with:

- Each edge, e , is assigned a positive real capacity, C_e .
- One (or more) vertex is labelled a **source**.
- One (or more) vertex is labelled a **sink**.

Example



Flows

- Next we want to be able to talk about flows (traffic) through a network.
- Rather than talking about where each car goes, we will instead concern ourselves with the total flow, f_e , through each edge e .

Flows

- Next we want to be able to talk about flows (traffic) through a network.
- Rather than talking about where each car goes, we will instead concern ourselves with the total flow, f_e , through each edge e .
- This must satisfy two conditions:

Rate Limitation

For each edge e ,

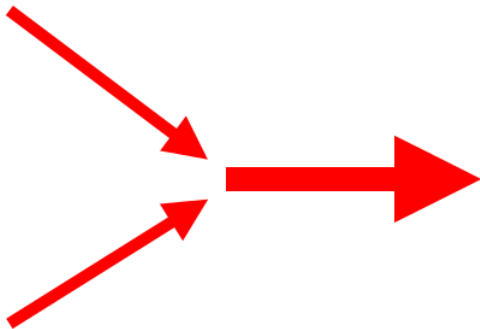
$$0 \leq f_e \leq C_e.$$



Conservation of Flow

For all v not a source or sink:

$$\sum_{e \text{ into } v} f_e = \sum_{e \text{ out of } v} f_e.$$



Formal Definition

Definition

A **flow** in a network is an assignment of a real number flow, f_e to each edge e so that

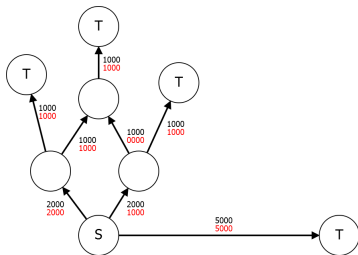
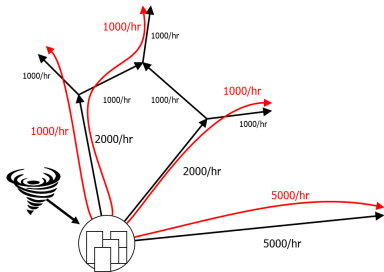
- For all e

$$0 \leq f_e \leq C_e.$$

- For all v not a source or sink

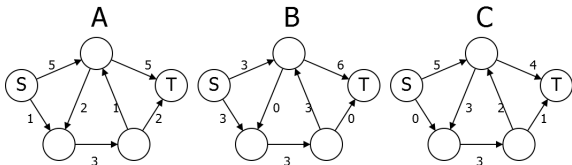
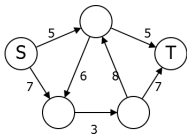
$$\sum_{e \text{ into } v} f_e = \sum_{e \text{ out of } v} f_e.$$

Example



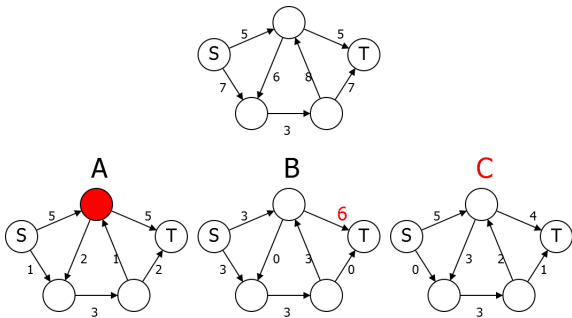
Problem

Which of the following is a valid flow for the given network?



Solution

Only flow **C** is valid. **A** fails to conserve flow at one vertex, and **B** exceeds.



Examples of Network Flows

Network flows are useful to study since they can model a number of real-life phenomena.

Flows of Goods on a Transportation Network



Flows of Electricity Through the Power Grid



Flows of Water Through Pipes



Flows of Information Through a Communications Network



Flow Size

One thing to know about a flow is how much stuff is actually flowing.

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One thing to know about a flow is how much stuff is actually flowing. This can be computed by looking at the sources.

Definition

For a flow, f , the **size** of the flow is given by

$$|f| := \sum_{e \text{ out of a source}} f_e - \sum_{e \text{ into a source}} f_e.$$

Sinks

You can also compute this by looking at sinks.

Lemma

$$|f| = \sum_{e \text{ into a sink}} f_e - \sum_{e \text{ out of a sink}} f_e.$$

Proof

$$\begin{aligned} 0 &= \sum_e f_e - \sum_e f_e \\ &= \sum_v \left(\sum_{e \text{ into } v} f_e - \sum_{e \text{ out of } v} f_e \right) \\ &= \sum_{v \text{ source or sink}} \left(\sum_{e \text{ into } v} f_e - \sum_{e \text{ out of } v} f_e \right) \end{aligned}$$

Proof

$$\begin{aligned} &= \sum_{v \text{ source or sink}} \left(\sum_{e \text{ into } v} f_e - \sum_{e \text{ out of } v} f_e \right) \\ &= \sum_{e \text{ into a source}} f_e + \sum_{e \text{ into a sink}} f_e \\ &\quad - \sum_{e \text{ out of a source}} f_e - \sum_{e \text{ out of a sink}} f_e \\ &= -|f| + \left(\sum_{e \text{ into a sink}} f_e - \sum_{e \text{ out of a sink}} f_e \right). \end{aligned}$$

Problem

How large a flow can we fit through a network?

Maxflow

Input: A network G

Output: A flow f for G with $|f|$ as large as possible.